stichting
mathematisch
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MC

AFDELING ZUIVERE WISKUNDE ZW 17/73 DECEMBER

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SCHNEIDER'S METHOD IN FIELDS OF CHARACTERISTIC $p \neq 0$

2e boerhaavestraat 49 amsterdam

Bibliotheek
Centrum voor Wiskunde en Informatica
Amsterdam

Printed at the Mathematical Centre, 49, 2e Boerhaavestraat, Amsterdam.

The Mathematical Centre, founded the 11-th of February 1946, is a non-profit institution aiming at the promotion of pure mathematics and its applications. It is sponsored by the Netherlands Government through the Netherlands Organization for the Advancement of Pure Research (Z.W.O), by the Municipality of Amsterdam, by the University of Amsterdam, by the Free University at Amsterdam, and by industries.

AMS (MOS) subject classification scheme (1970): 10F35

Schneider's method in fields of characteristic $p \neq 0$

by

J.M. Geijsel

Abstract

In this report two theorems on transcendental elements of fields of characteristic $p > 0$, namely L.I. Wade's result on the analogue of the Gelfond-Schneider theorem for fields of characteristic p (see Duke Math. J. 13 (1946), 79-85), and my result on the trancendency of certain values of the Carlitz-Bessel functions (see Math. Centre Report ZW 2/71, Amsterdam) are generalized for a wider class of so called E-functions.

Let $\mathbb{F}_q$ be a finite field of characteristic $p \neq 0$ with $p = q^{n_0}$ elements. We denote by $\mathbb{F}_q[x]$ the ring of polynomials with coefficients in $\mathbb{F}_q$ and by $\mathbb{F}_q\{x\}$ its quotientfield.

For $0 \neq E \in \mathbb{F}_q[x]$ we define the (logarithmic) valuation

$$\mathrm{dg}\, E = \text{degree of } E \text{ and } \mathrm{dg}\, 0 = -\infty.$$

For $Q \in \mathbb{F}_q\{x\}$ where $Q = \frac{E}{F}$ with $E,F \in \mathbb{F}_q[x]$ and $F \neq 0$ we define

$$\mathrm{dg}\, Q = \mathrm{dg}\, E - \mathrm{dg}\, F.$$

The completion of $\mathbb{F}_q\{x\}$ with respect to the valuation is denoted by F and the completion of the algebraic closure of F by Φ. The valuation dg on $\mathbb{F}_q\{x\}$ can be extended to Φ in a unique way and also will be denoted by dg.

A function $f : \Phi \to \Phi$ given by a power series

$$f(t) = \sum_{i=0}^{\infty} a_i\, t^i \quad \text{with } a_i \in \Phi,$$

which converges for all t with $\mathrm{dg}\, t < R$ is called *linear* if

$$\begin{cases} f(t+u) = f(t) + f(u) & \forall t,u \in \Phi \text{ with } \mathrm{dg}\, t < R,\ \mathrm{dg}\, u < R, \\ f(ct) = cf(t) & \forall t \in \Phi \text{ with } \mathrm{dg}\, t < R \text{ and } c \in \mathbb{F}_q. \end{cases}$$

For linear functions we define for all t for which the involving series converge the operators Δ^r $(r=1,2,\dots)$ by

$$\Delta f(t) = f(xt) - x\, f(t),$$
$$\Delta^r f(t) = \Delta^{r-1} f(xt) - x^{q^{r-1}}\, \Delta^{r-1} f(t), \qquad r \geq 2.$$

For purpose of notation we define $\Delta^0 f(t) = f(t)$.

A function $f : \Phi \to \Phi$ is said to be *entire* if f can be written as a power series with coefficients in Φ, which converges for all $t \in \Phi$.

For entire linear functions f we have an "expansion formula" (see [1], or [2] lemma 2.1), namely :

for every $M \in \mathbb{F}_q[x]$ we have

$$f(Mt) = \sum_{\nu=0}^{dg\,M} \frac{\psi_\nu(M)}{F_\nu} \Delta^\nu f(t),$$

where

$$F_\nu := (x^{q^\nu}-x)(x^{q^\nu}-x^q) \dots (x^{q^\nu}-x^{q^{\nu-1}}), \quad \nu \geq 1$$

$$F_0 := 1$$

$$\psi_\nu(t) := \prod_{\substack{dg\ E<\nu \\ E\in\mathbb{F}_q[x]}} (t-E).$$

Now we introduce a special class of linear functions.

Definition. A linear function $f : \Phi \to \Phi$ given by

$$f(t) = \sum_{k=0}^{\infty} a_k \frac{t^{q^k}}{F_k}$$

is called an *E-function* if there exists a finite separable algebraic extension K of F of degree h such that:

(1) $a_k \in K$, $k = 0,1,\dots$

(2) $\exists c \in \mathbb{R}$, $c > 0$ such that $dg\ a_k < cq^k$

(3) $\forall k \in \mathbb{N} \cup \{0\}$ $\exists Q_k \in \mathbb{F}_q[x]$ of minimal degree such that $Q_k a_0, Q_k a_1, \dots, Q_k a_k$ are integers in K and $dg\ Q_k = O(kq^k)$, $k \to \infty$.

Remarks

(i) From (1) and (2) we have that every E-function is entire.

(ii) The functions $\psi(t)$ and $J_n(t)$ (see [3]) are E-functions.

(iii) Linear polynomials with separable algebraic coefficients in Φ are E-functions.

(iv) If f and g are E-functions then $\Delta^r f$ $(r\geq 1)$, f^{q^r}, $f + g$ are E-functions.

(v) If P is a linear polynomial with separable algebraic coefficients in Φ and f is an E-function, then $P(f(t))$ is an E-function.

Lemma 1. Let K be a separable finite algebraic extension of $\mathbb{F}\{x\}$ of degree h. Let $r, s \in \mathbb{N}$ with $0 < r < s$. Then the system of linear equations

$$\sum_{i=1}^{s} \alpha_{ki} X_i = 0 \qquad (k=1,\dots,r),$$

where α_{ki} are algebraic integers in K and

$$a = \max_{k,i} (\mathrm{dg}\, \alpha_{ki}, 0)$$

has a non-trivial solution $\{X_i\}_{i=1}^{s}$ with

$$X_i \in \mathbb{F}_q[x]$$

such that

$$\mathrm{dg}\, X_i < \frac{cs + ar}{s - r} \qquad (i=1,\dots,s),$$

where c is a positive constant only depending on the field K.

Proof. We use the following lemma which will be proved in an appendix.

Lemma: Let K be a separable finite algebraic extension of $\mathbb{F}_q\{x\}$ of degree h. Then there exists a basis $\beta_1, \dots, \beta_h$ of algebraic integers of K such that every algebraic integer $\xi \in K$ can be written uniquely as

$$\xi = \sum_{i=1}^{h} A_i \beta_i \quad \text{with} \quad A_i \in \mathbb{F}_q[x].$$

Further we use the methods of lemma 4.2 in [3]. □

Now we can formulate the main result of this paper.

Theorem 1. Let $f_1(t), \dots, f_n(t)$ be $\mathcal{E}$-functions, not all polynomials. Suppose

$$\Delta^r f_\nu(t) = R_{\nu r}(f_1(t),\dots,f_n(t)) \quad , \quad r = 0,1,\dots; \ \nu = 1,\dots,n,$$

where $R_{\nu r}$ are n-linear polynomials of n variables $f_1, \dots, f_n$ of total degree $<q^r$ with coefficients in $\mathbb{F}_q[x]$ of degree $<q^r$.

Let $\alpha \neq 0$, $\beta \notin \mathbb{F}_q\{x\}$ and $f_\nu(t) \neq 0$. Then at least one of the elements

$$\{\beta , f_1(\alpha) , \dots , f_n(\alpha) , f_1(\alpha\beta) , \dots , f_n(\alpha\beta)\}$$

is transcendental over $\mathbb{F}_q\{x\}$.

Corollary 1.

a) With the choice $f_1(t) = \psi(t)$, $\alpha = \lambda(\alpha^*)$, where α^* is not a zero of $\lambda(t)$, and $\beta \notin \mathbb{F}_q\{x\}$ we get the analogue of the theorem of Gelfond-Schneider: at least one of the elements $\{\beta , \alpha^* = \psi(\lambda(\alpha^*)) , \psi(\beta\lambda(\alpha^*))\}$ is transcendental over $\mathbb{F}_q\{x\}$. This result was proved by Wade in [5].

b) With $f_1(t) = J_n(t)$, $f_2(t) = \Delta J_n(t)$ and $\alpha \neq 0$, $\beta \notin \mathbb{F}_q\{x\}$ we get: at least one of the elements

$$\{\beta , J_n(\alpha) , \Delta J_n(\alpha) , J_n(\alpha\beta) , \Delta J_n(\alpha\beta)\}$$

is transcendental over $\mathbb{F}_q\{x\}$. This result was essentially proved in [3], where the theorem said under the same conditions for α and β: at least one element of the set $V = \{\alpha , \beta , J_n(\alpha) , \Delta J_n(\alpha) , J_n(\alpha\beta) , \Delta J_n(\alpha\beta)\}$ is transcendental over $\mathbb{F}_q\{x\}$. At the begin of the proof in [3] we supposed α to be algebraic over $\mathbb{F}_q\{x\}$ but we didn't use this fact, hence α can be omitted in V.

Corollary 2. If we choose $f_1(t) = \psi(\alpha_1^* t), \dots, f_n(t) = \psi(\alpha_n^* t)$, where $\alpha_\nu^* \neq 0$, $\nu=1,\dots,n$, and if $\alpha = 1$, $\beta \notin \mathbb{F}_q\{x\}$ then at least one of the elements $\{\beta, \psi(\alpha_1^*),\dots,\psi(\alpha_n^*) , \psi(\alpha_1^*\beta),\dots,\psi(\alpha_n^*\beta)\}$ is transcendental over $\mathbb{F}_q\{x\}$.

If now α_i is not a zero of $\lambda(t)$, $i=1,\dots,n$, and $\alpha_i^* := \lambda(\alpha_i)$, then at least one of the elements of the set $\{\beta,\alpha_1,\dots,\alpha_n,\psi(\beta\lambda(\alpha_1)),\dots \dots,\psi(\beta\lambda(\alpha_n))\}$ is transcendental over $\mathbb{F}_q\{x\}$. For $n = 1$ we have the result of corollary 1a.

For $\beta \notin \mathbb{F}_q\{x\}$, α algebraic and $\lambda(\alpha) \neq 0$ we have: at least one of the elements $\{\beta, \psi(\beta\lambda(\alpha)),\psi(\beta\lambda(\alpha^q),\dots , \psi(\beta\lambda(\alpha^{q^n}))\}$, $n \geq 1$, is transcendental over $F_q\{x\}$. Equivalent with this last result is: at least one of the elements $\{\beta, \psi(\lambda(\alpha)), \psi(\beta\lambda(x\alpha)), \ldots, \psi(\beta\lambda(x^n\alpha))\}$, $n \geq 1$, is transcendental over $\mathbb{F}_q\{x\}$, since $\Delta\lambda(t) = \lambda(xt) - x\lambda(t) = \lambda(t^q)$.

Proof of theorem 1. Suppose $\beta, f_1(\alpha),\ldots,f_n(\alpha),f_1(\alpha\beta),\ldots,f_n(\alpha\beta)$ are algebraic over $\mathbb{F}_q\{x\}$, then, for some $e \in \mathbb{N}$, $\beta^{q^e}, f_1^{q^e}(\alpha),\ldots, f_n^{q^e}(\alpha)$, $f_1^{q^e}(\alpha\beta),\ldots, f_n^{q^e}(\alpha\beta)$ are separable over $\mathbb{F}_q\{x\}$ and they generate a separable extension K of $\mathbb{F}_q\{x\}$ of degree h.

Let $\Gamma \in \mathbb{F}_q[x]$ be such that $\Gamma\beta^{q^e}$, $\Gamma f_\nu^{q^e}(\alpha)$, $\Gamma f_\nu^{q^e}(\alpha\beta)$, $\nu=1,\ldots,n$, are algebraic integers of K.

The natural numbers k, ℓ with $k < \frac{1}{3}\ell$ will be chosen later.

Define

$$L(t) := \sum_{\nu=1}^{n} \sum_{j=0}^{q^{2\ell}-1} \sum_{i=0}^{q^{2k}-1} X_{ij\nu}\, t^{jq^e} f_\nu^{iq^e}(\alpha t),$$

where the polynomials $X_{ij\nu}$ will be determined by the following:

$$L(A+\beta B) = 0 \quad \text{for all } A,B \in \mathbb{F}_q[x] \text{ with } \mathrm{dg}\, A < m,\ \mathrm{dg}\, B < m,$$

$$\text{where } m := k + \ell - 1.$$

Since $\beta \notin \mathbb{F}_q\{x\}$ we get a linear system of at most q^{2m} equations in $nq^{2k+2\ell}$ variables $X_{ij\nu}$ with algebraic coefficients:

$$(1)\quad L(A+\beta B) = \sum_{\nu=1}^{n} \sum_{j=0}^{q^{2\ell}-1} \sum_{i=0}^{q^{2k}-1} X_{ij\nu}\, (A+\beta B)^{jq^e} f_\nu^{iq^e}(\alpha(A+\beta B)) = 0,$$

$$\mathrm{dg}\, A < m,\ \mathrm{dg}\, B < m.$$

$f_i(\alpha A+\alpha\beta B) = f_i(\alpha A) + f_i(\alpha\beta B)$ since f_i is linear. Since

$$\Delta^\mu f_i(t) = R_{i\mu}(f_1(t),\ldots,f_n(t)) =$$

$$=: \sum_{0\leq j_1+\ldots+j_n\leq\mu} A_{i\mu j_1\ldots j_n} f_1^{q^{j_1}}(t)\ldots f_n^{q^{j_n}}(t)$$

with $A_{i\mu j_1\ldots j_n} \in \mathbb{F}_q[x]$ and $\mathrm{dg}\, A_{i\mu j_1\ldots j_n} < q^{\mu}$, the expansion formula gives:

$$f_i(\alpha A) = \sum_{\mu=0}^{\mathrm{dg}\, A} \frac{\psi_\mu(A)}{F_\mu} \sum_{j_1+\ldots+j_n\le\mu} A_{i\mu j_1\ldots j_n} f_1^{q^{j_1}}(\alpha)\ldots f_n^{q^{j_n}}(\alpha).$$

Since $\mathrm{dg}\frac{\psi_\mu(A)}{F_\mu} \le \max\limits_{0\le\mu\le\mathrm{dg}\, A} \mathrm{dg}\frac{\psi_\mu(A)}{F_\mu} = \max\limits_{0\le\mu\le\mathrm{dg}\, A} (q^\mu \mathrm{dg}\, A - \mu q^\mu) \le \mathrm{dg}\, A \,.\, q^{\mathrm{dg}\, A}$,

$$\mathrm{dg}\, f_i(\alpha A) \le mq^m + q^m + q^m \max\{\mathrm{dg}\, f_1(\alpha),\ldots,\mathrm{dg}\, f_n(\alpha),0\}.$$

Since $f_i^{q^e}(\alpha A)$ resp. $f_i^{q^e}(\alpha\beta A)$ is a polynomial in $f_\nu^{q^e}(\alpha)$ resp. $f_\nu^{q^e}(\alpha\beta)$, $i,\nu \in (1,\ldots,n)$, of degree $\le q^m$ with coefficients in $\mathbb{F}_q\{x\}$, the coefficients of $X_{ij\nu}$ in (1) are polynomials in

$$\beta^{q^e} \text{ of degree } \le q^{2\ell} - 1$$

$$f_\nu^{q^e}(\alpha)\ ,\ f_\nu^{q^e}(\alpha\beta) \quad \text{of degree} \le (q^{2k}-1)q^m$$

with coefficients in $\mathbb{F}_q\{x\}$.

Since $q^{2\ell} - 1 + 2n(q^{2k}-1)q^m < q^{2\ell+2n}$ we can get a system of equations with integral algebraic coefficients in K by multiplying each equation with the factor

$$\Gamma^{q^{2\ell+2n}} (F_m^{q^e})^{q^{2k}-1}.$$

This gives the system of equations:

$$\Gamma^{q^{2\ell+2n}} (F_m^{q^e})^{q^{2k}-1} L(A+\beta B) = 0 \qquad \text{for } \mathrm{dg}\, A, \mathrm{dg}\, B < m;\ A,B \in \mathbb{F}_q[x]$$

which we denote by

$$(2) \quad \sum_{\nu=1}^{n} \sum_{j=0}^{q^{2\ell}-1} \sum_{i=0}^{q^{2k}-1} X_{ij\nu} D_{ij\nu} = 0 \quad \text{for } A,B \in \mathbb{F}_q[x];\ \mathrm{dg}\, A, \mathrm{dg}\, B < m.$$

Since $m = k + \ell - 1$ the number of equations, q^{2m}, is less than the number of variables $n\, q^{2k+2\ell}$. Furthermore

$$\mathrm{dg}\ D_{ij} \le q^{2\ell+2n}\ \mathrm{dg}\ \Gamma + mq^{m+2k+e} + q^{2\ell+e}(m+c_1) + q^{2k+e}(mq^m+q^m+q^m c_0)$$

where $c_1 = \max(\mathrm{dg}\beta, 0)$; $c_0 = \max(\mathrm{dg}\ f_\nu(\alpha);\ \mathrm{dg}\ f_\nu(\alpha\beta),\ (\nu=1,\dots,n);\ 0)$, which gives (since $k < \frac{1}{3}\ell$):

$$\mathrm{dg}\ D_{ij\nu} \le (3m+c_2)q^{2\ell+e} \qquad \text{where } c_2 \ge 0.$$

According to lemma 1 with $r = q^{2m}$, $s = nq^{2k+2\ell}$ and $a = \max\limits_{i,j,\nu}(\mathrm{dg}\ D_{ij\nu}, 0)$ we have that there exist polynomials $X_{ij\nu} \in \mathbb{F}_q[x]$, not all zero, such that (1) is satisfied and

(3) $\quad \mathrm{dg}\ X_{ij\nu} \le (3m+c_3)q^{2\ell+e} \qquad \text{where } c_3 \ge 0.$

Now we shall prove that, for all $A, B \in \mathbb{F}_q[x]$, $L(A+\beta B) = 0$. Let $\mu \ge m$ and $\eta = \mu - k + 1$, then $\eta \ge \ell$. Furthermore let

$$\mathcal{B}(\mu) = \{A + \beta B \mid \mathrm{dg}\ A < \mu,\ \mathrm{dg}\ B < \mu,\ A \text{ and } B \text{ not both } 0\}.$$

Suppose $L(t) = 0$ for all $t \in \mathcal{B}(\mu)$. Let $\xi \in \mathcal{B}(\mu+1) \setminus \mathcal{B}(\mu)$, then $\mathrm{dg}\ \xi = \mu + g$ with $g \ge 0$. We choose ℓ such that $m + g < 2m$. By assumption

$$L(t) \Big/ \prod_{\mathcal{B}(\mu)} (t-A-\beta B)$$

is an entire function since $L(t)$ is entire. According to the maximum-modulus-principle

$$\mathrm{dg}\left(\frac{L(\xi)}{\prod_{\mathcal{B}(\mu)}(\xi-A-\beta B)}\right) \le \max_{\mathrm{dg}\,t=2\mu} \mathrm{dg}\left(\frac{L(t)}{\prod_{\mathcal{B}(\mu)}(t-A-\beta B)}\right).$$

Hence

$$\mathrm{dg}\ L(\xi) \le \max_{\mathrm{dg}\,t=2\mu} \mathrm{dg}\ L(t) - 2\mu(q^{2\mu}-1) + (\mu+g)(q^{2\mu}-1).$$

From the definition of $L(t)$ we get

$$\max_{\mathrm{dg}\,t=2\mu} \mathrm{dg}\ L(t) \le \max_{i,j,\nu} \mathrm{dg}\ X_{ij\nu} + 2\mu q^{2\ell+e} + q^{2k+e} \max_{\mathrm{dg}\,t=2\mu} \mathrm{dg}\ f_\nu(\alpha t).$$

Since f_ν is an E-function we have

$$f_\nu(t) = \sum_{k=0}^{\infty} a_{\nu k} \frac{t^{q^k}}{F_k}, \qquad \nu = 1,\dots,n,$$

where $\exists c > 0$ such that $\deg a_{\nu k} < cq^k$ for $k > k_0$ and $\nu = 1,\dots,n$. Hence

$$\max_{\deg t = 2\mu} \deg f_\nu(\alpha t) \le \max_{k \ge 0} (\deg a_{\nu k} + 2\mu q^k - kq^k + q^k \deg \alpha) <$$

$$< \max_{k \ge 0} (c + 2\mu - k + \deg \alpha) q^k \le c_4 \, q^{2\mu}$$

where $c_4 > 0$ and $\mu \ge m$. This gives

$$\deg L(\xi) \le (3m + c_3) q^{2\ell + e} + 2\mu q^{2\ell + e} + c_4 q^{2\mu + 2k + e} - (\mu - g) q^{2\mu}.$$

Since $\mu \ge m$, $\eta \ge \ell$ and $\mu = \eta + k - 1$ we get

$$(4) \quad \deg L(\xi) \le q^{2\eta + e} \, (5\mu + c_5 q^{4k} - (\mu - g) q^{2k - e}).$$

$L(\xi)$ is a polynomial in β^{q^e} of degree $q^{2\ell} - 1$ and in each of the $f_\nu^{q^e}(\alpha)$, $f_\nu^{q^e}(\alpha\beta)$ of degree $(q^{2k} - 1)q^\mu$, hence $L(\xi)$ is algebraic; since

$$q^{2\ell} + 2nq^{2k+\mu} < q^{2\eta + 2n},$$

$F_\mu^{q^{2k+e}} \; \Gamma^{q^{2\eta+2n}} \; L(\xi)$ is an algebraic integer of K. Therefore

$$N(F_\mu^{q^{2k+e}} \; \Gamma^{q^{2\eta+2n}} \; L(\xi)) \in \mathbb{F}_q[x]$$

and

$$\deg N(F_\mu^{q^{2k+e}} \; \Gamma^{q^{2\eta+2e}} \; L(\xi)) \le$$

$$\le h[\mu q^{2k+e+\mu} + q^{2\eta+2n} \deg \Gamma + q^{2\eta+e} \, (5\mu + c_5 q^{4k} - (\mu - g) q^{2k-e})] \le$$

$$\le h \, q^{2\eta+e} \, [(6\mu + c_6 q^{4k}) - (\mu - g) q^{2k-e}],$$

which is negative for sufficiently large k and ℓ. Now choose k and ℓ such that dg $N(F_\mu^{q^{2k+e}} \Gamma^{q^{2\eta+2e}} L(\xi))$ is negative. Hence $L(\xi) = 0$.

In the following k and ℓ are fixed. Since $\beta \notin \mathbb{F}_q\{x\}$ all $A + \beta B$ are different and L(t) has an infinite number of zero's. Since L(t) is entire and not a polynomial L(t) is a transcendental function (see [4] or [2]).

Let N be the set of zero's $\neq 0$ of L(t), then N is countable and for any $\nu \in \mathbb{N}$

$$L(t) = \gamma_0 t^\rho \prod_{\xi \in B(\nu)} (1 - \frac{t}{\xi}) \prod_{\substack{\xi \notin B(\nu) \\ \xi \in N}} (1 - \frac{t}{\xi}) \quad \text{with } \rho \geq 0 \text{ and } \gamma_0 \in \Phi.$$

Let ν_0 be the minimum of the degrees of the zero's $\neq 0$ of L(t), then

$$\max_{\mathrm{dg}\, t = 2\nu} \mathrm{dg} \prod_{\xi \in N \setminus B(\nu)} (1 - \frac{t}{\xi}) \geq \max_{\mathrm{dg}\, t = \frac{\nu_0}{2}} \mathrm{dg} \prod_{\xi \in N \setminus B(\nu)} (1 - \frac{t}{\xi}) = 0.$$

Furthermore

$$\prod_{\xi \in B(\nu)} (1 - \frac{t}{\xi}) = \frac{\prod_{B(\nu)} (A+\beta B-t)}{\prod_{B(\nu)} (A+\beta B)},$$

hence

$$\max_{\mathrm{dg}\, t = 2\nu} \mathrm{dg}\, L(t) \geq c_7 + 2\nu\rho + 2\nu(q^{2\nu}-1) - (\nu+g)(q^{2\nu}-1),$$

where c_7 is a constant only depending on L(t). This gives

$$(5) \qquad \max_{\mathrm{dg}\, t = 2\nu} \mathrm{dg}\, L(t) \geq (c_8\nu + c_9) q^{2\nu} \quad \text{with } c_8 > 0.$$

On the other hand we have proved

$$(6) \qquad \max_{\mathrm{dg}\, t = 2\nu} \mathrm{dg}\, L(t) \leq (3m+c_3) q^{2\ell+e} + 2\nu q^{2\ell+e} + c_4 q^{2\nu+2k+e}.$$

For ν large enough (5) and (6) are contradictory, which completes the proof of the theorem. □

Remark. The theorem is also true for systems $\{f_1,\ldots,f_n\}$ for which the following relation is true:

$$\Delta^2 f_\nu(t) = R_{\nu r}(f_1(t),\ldots,f_n(t)), \quad r=0,1,\ldots; \ \nu=1,2,\ldots,n,$$

where

$$R_{\nu r}(f_1(t),\ldots,f_n(t)) = \sum_{j_1+\ldots+j_n\le r} Q_{\nu r j_1\ldots j_r} f_1^{q^{j_1}}(t)\ldots f_n^{q^{j_n}}(t)$$

with $Q_{\nu r j_1\ldots j_r} \in \mathbb{F}_q\{x\}$, such that for all $r \ge 0$ there exists a polynomial A_r such that $Q_{\nu\rho j_1\ldots j_r} A_r \in \mathbb{F}_q[x]$ for $\rho=0,1,\ldots,r$; $1 \le \nu \le n$; $j_1+\ldots+j_n \le r$ and dg $A_r < q^r$.

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APPENDIX

Lemma. Let K be a separable finite algebraic extension of $\mathbb{F}_q\{x\}$ of degree h. Then there exists a basis $\beta_1, \ldots, \beta_h$ of algebraic integers of K such that every algebraic integer $\xi \in K$ can be written uniquely as

$$\xi = \sum_{i=1}^{h} A_i \beta_i \quad \text{with } A_i \in \mathbb{F}_q[x].$$

Proof. According to the theorem of the primitive element [*], since K is a separable finite extension of $\mathbb{F}_q\{x\}$ of degree h there exists an element $\theta \in K$ such that $K = \mathbb{F}_q\{x\}(\theta)$. θ is a separable algebraic element of K, hence there is a polynomial P in $\mathbb{F}_q[x]$ such that $P\theta$ is an algebraic integer of K. Denote $P\theta$ again by θ. Let $\theta_1 = \theta, \theta_2, \ldots, \theta_h$ be the conjugate elements of the algebraic integer θ. The discriminant $\Delta(1,\theta,\ldots,\theta^{h-1})$ of the basis $1,\theta,\ldots,\theta^{h-1}$ of $K / \mathbb{F}_q\{x\}$ is a Van der Monde determinant and since θ is separable, $\theta_i \neq \theta_j (i \neq j)$; hence $\Delta(1,\theta,\ldots,\theta^{h-1}) \neq 0$. Furthermore is

$$\Delta(1,\theta,\ldots,\theta^{h-1}) = \prod_{1 \leq i<j \leq h} (\theta_i - \theta_j)^2$$

a symmetric polynomial in the conjugate elements of θ and can be expressed as a polynomial in the coefficients of the minimal polynomial of θ; hence $\Delta(1,\theta,\ldots,\theta^{h-1}) \in \mathbb{F}_q[x]$.

For every base $\{w_1, \ldots w_h\}$ of $K / \mathbb{F}_q\{x\}$ with w_i algebraic integer in K we have

$$\Delta(w_1,\ldots,w_h) = (\det(a_{ij}))^2 \cdot \Delta(1,\theta,\ldots,\theta^{h-1})$$

where $w_i = a_{i1} + a_{i2}\theta + \ldots + a_{ih}\theta^{h-1}$ $(i=1,\ldots,h)$ with $a_{ij} \in \mathbb{F}_q\{x\}$, and $\det a_{ij} \neq 0$. On the other hand as a symmetric polynomial in the algebraic integers $w_1, \ldots, w_h$ and its conjugates $\Delta(w_1,\ldots,w_h) \in \mathbb{F}_q[x]$. Consider all bases $\{w_1,\ldots,w_h\}$ for $K / \mathbb{F}_q\{x\}$ with algebraic integers $w_1, \ldots, w_h$. Then $dg\ \Delta(w_1,\ldots,w_h) \in \mathbb{N} + \{0\}$, hence there exists a basis $\{\beta_1,\ldots,\beta_h\}$ with $dg\ \Delta(\beta_1,\ldots,\beta_h)$ minimal and $\beta_1, \ldots, \beta_h$ algebraic integers. We shall prove that this basis $\{\beta_1,\ldots,\beta_h\}$ is a basis for the ring of algebraic integers in K over $\mathbb{F}_q[x]$.

Suppose $\{\beta_1,\dots,\beta_h\}$ is not a basis for the integers in K over $\mathbb{F}_q[x]$, then there exists an algebraic integer $\xi \in K$ such that $\xi = a_1\beta_1+\dots+a_h\beta_h$ with $a_i \in \mathbb{F}_q\{x\}$ and not all $a_i \in \mathbb{F}_q[x]$. Suppose $a_1 \notin \mathbb{F}_q[x]$. $a_1 = A + r$ with $A \in \mathbb{F}_q[x]$ and $r \in \mathbb{F}_q\{x\} \setminus \mathbb{F}_q[x]$, dg $r < 0$ and $r \neq 0$. Now define

$$\beta_1^* = \xi - A\beta_1 = (a_1-a)\beta_1 + a_2\beta_2 + \dots + a_h\beta_h$$
$$\beta_i^* = \beta_i\,, \qquad i=2,\dots,h.$$

The system $\{\beta_1^*,\dots,\beta_h^*\}$ is a basis for $K / \mathbb{F}_q\{x\}$ and β_i^* $(i=1,\dots,h)$ are algebraic integers.

$$\Delta(\beta_1^*,\dots,\beta_h^*) = \det \begin{pmatrix} a_1-a & a_2 & a_3 & \dots & a_h \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \cdot & \cdot & \cdot & \dots & \cdot \\ \cdot & \cdot & \cdot & \dots & \cdot \\ 0 & 0 & 0 & & 1 \end{pmatrix}^2 \Delta(\beta_1,\dots,\beta_h)$$
$$= r^2\,\Delta(\beta_1,\dots,\beta_h).$$

$$\text{dg}\ \Delta(\beta_1^*,\dots,\beta_h^*) = 2\ \text{dg}\ r + \text{dg}\ \Delta(\beta_1,\dots,\beta_h) < \text{dg}\ (\beta_1,\dots,\beta_h).$$

This contradicts the minimality of dg $\Delta(\beta_1,\dots,\beta_h)$ and proves the lemma. □

[*] B.L. van der Waerden, *Algebra* I, §43.